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Quantum theory of the relativistic oscillator from the HOOP (higher-order one-particle) viewpoint

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Abstract. Expanding on earlier work, it is shown how to rewrite the non-relativistic problem of two particles coupled by a spring as one higher-order one-particle (HOOP) problem, which then can be elevated to be Lorentz covariant. The higher-order Lagrangian is then developed into an Ostrogradsky Hamiltonian format, from which canonical commutation rules are invoked and a relativistic wave equation produced. Within the centre-of-momentum frame of reference, the latter is reduced to a radial Schrödinger equation that is solved numerically, with a display of radial eigenfunctions that are compared with non-relativistic ones. The details of this example illustrate conclusively how the HOOP approach to direct-interaction many-particle relativistic dynamics works.

1. Introduction

The theory of directly interacting relativistic particles has defied adequate exposition for half a century, notwithstanding strong efforts to elaborate upon it. The subject is reviewed in several monographs and papers [1, 2]. Its attractiveness is that it attempts to broach relativistic interaction quite directly, in a finite and non-perturbative physical set-up referring only to the orbits of particles, without the infinities of mediating fields. The difficulty since the beginning has been the zero-interaction theorem (ZIT) of Currie *et al* [3] which states that if the physical coordinates of particles running on invariant worldlines are taken as canonical coordinates in a Hamiltonian dynamics, while the Poincaré group is canonically represented, then the only possible orbits are those for free non-interacting particles.

In this paper a previously devised scheme [4] is worked out in detail for overcoming the ZIT, using the primitive example of a pair of particles hooked together by an ideal spring (in the non-relativistic limit) and carried through to a complete quantum-mechanical elaboration.

The idea is very simple. What drives the ZIT is the non-covariance of simultaneity in special relativity, as depicted in figure 1 where the schematic worldlines of a pair of interacting particles are shown with respect to a frame x, ct and a boosted frame x', ct'. The shift to new simultaneity from B to B' along the worldline of particle 2 produces, infinitesimally, a shift in the action $L(x_1 x_2 \dot{x}_1 \dot{x}_2)\,\mathrm{d}t$ which, if it is to be at most an exact differential that secures the boost to be canonically represented, drives L into the form of a sum of free-particle Lagrangians, one for each particle, with no coupling between them.

In fact figure 1 contains a significant redundancy in the following sense: if the two equations of motion for $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ are decoupled, giving one single equation of motion, say for $\boldsymbol{x}_1(t)$ alone, of fourth order, than the worldline of particle 1 alone may be plotted as in

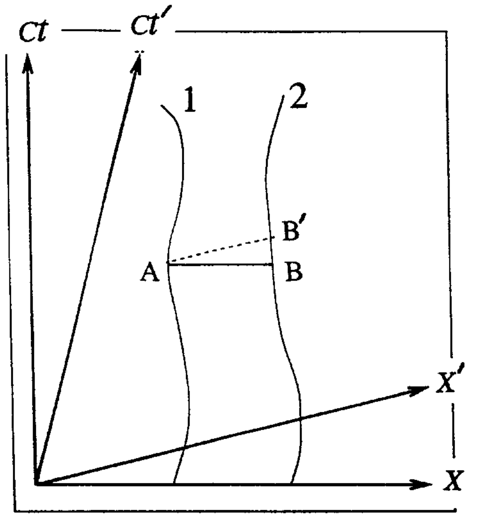

Figure 1. Non-covariance of simultaneity in many-particle dynamics, requiring a shift from $\boldsymbol{B}$ to $\boldsymbol{B}'$ under a boost.

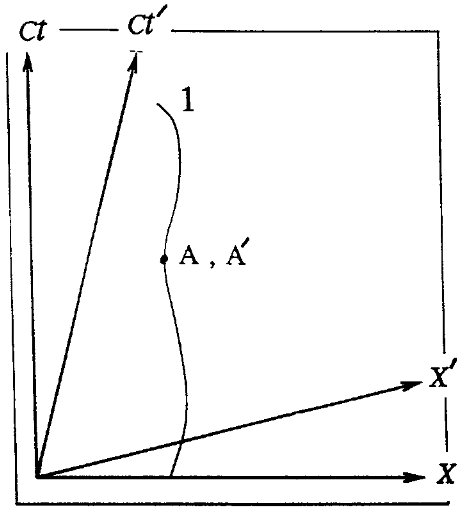

Figure 2. With the motion of particle 2 eliminated according to HOOP, just the orbit of particle 1 alone suffices in one higher-order differential equation to span the two-particle dynamics. The boost now requires no shift along a worldline, but merely a kinematical redesignation of one and the same worldpoint as A in (x, ct) or A′ in (x', ct').

figure 2 and spans perfectly the entire physical situation of an interacting pair of particles, no more and no less. This means that a complete and completely self-referential description of *one* worldline, bought at the cost of an elevation of differential order, suffices fully to describe many-particle dynamics.

This higher-order one-particle (HOOP) representation of dynamics has, in itself, nothing to do with relativity, but merely states that a pair of second-order equations of motion in two variables is equivalent to one fourth-order equation in one of them. The argument extends in principle to the n-particle problem,

$$\ddot{\boldsymbol{r}}_i = f_i(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n, \dot{\boldsymbol{r}}_1\dot{\boldsymbol{r}}_2, \ldots, \dot{\boldsymbol{r}}_n) \qquad i = 1, 2, \ldots, n$$

where, by an ancient prescription, repeated differentiation of one equation of motion say $\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1$ and elimination of $\boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_n, \dot{\boldsymbol{r}}_2\dot{\boldsymbol{r}}_3, \ldots, \dot{\boldsymbol{r}}_n$ eventually brings

$$\boldsymbol{r}_1^{(2n)} = F(\boldsymbol{r}_1, \dot{\boldsymbol{r}}_1, \ldots, \boldsymbol{r}_1^{(2n-1)}).$$

Thus, for example, in the two-particle problem one must invert $\ddot{\boldsymbol{r}}_1$ and $\dddot{\boldsymbol{r}}_1$ as functions of $\boldsymbol{r}_1, \dot{\boldsymbol{r}}_1, \boldsymbol{r}_2, \dot{\boldsymbol{r}}_2$ to give $\boldsymbol{r}_2$ and $\dot{\boldsymbol{r}}_2$ as functions of $\boldsymbol{r}_1, \dot{\boldsymbol{r}}_1, \ddot{\boldsymbol{r}}_1, \dddot{\boldsymbol{r}}_1$, which may be difficult technically. However, it is readily executed for a considerable family of central forces [4].

The crucial point specifically regarding relativity is now that the troublesome shift to new simultaneity under a boost (like $\boldsymbol{B}$ to $\boldsymbol{B}'$ in figure 1) is eliminated in favour of a pointwise redescription of the single orbit without shifting along the worldline, and the description of the single orbit may then be made manifestly Lorentz covariant without difficulty.

A procedure, long sought, which elevates via HOOP a Galilean covariant dynamics into a Poincaré covariant dynamics using the two-particle case initially in $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, as a prototype is the following. Firstly, working non-relativistically from the Newtonian equations of motion, decouple them by computing $\dddot{\boldsymbol{r}}_1$ and $\boldsymbol{r}_1^{(4)}$ from $\ddot{\boldsymbol{r}}_1$ and eliminating $\boldsymbol{r}_2$ and $\dot{\boldsymbol{r}}_2$ to obtain one fourth-order equation in $\boldsymbol{r}_1$, with all reference to particle 2 entirely removed. The Galilean covariance of the starting pair will be carried over into Galilean covariance of the single fourth-order HOOP equation of motion for particle 1. Secondly, develop a higher-order Lagrangian L_0—in $\boldsymbol{r}_1$, $\dot{\boldsymbol{r}}_1$, $\ddot{\boldsymbol{r}}_1$—for fourth-order HOOP, producing its Euler–Lagrange equation of motion as follows:

$$\frac{\partial L_0}{\partial \boldsymbol{r}_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \dot{\boldsymbol{r}}_1} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L_0}{\partial \ddot{\boldsymbol{r}}_1} = 0.$$

This will ordinarily be feasible if the starting pair itself possessed a Lagrangian. Thirdly and critically, step up the action $L_0\,\mathrm{d}t$ into a Lorentz scalar $L\,\mathrm{d}\tau$ by introducing the proper time interval $\mathrm{d}\tau$ and suitable Lorentz scalars like $\boldsymbol{a}_\mu\boldsymbol{a}_\mu$ (where $\boldsymbol{a}_\mu$ denotes 4-acceleration) to generalize the Newtonian squared-acceleration $\boldsymbol{a}^2$, in such a way as to recover the Newtonian action $L_0\,\mathrm{d}t$ in a the non-relativistic limit. This elevation from Galilean covariance to Poincaré covariance is not necessarily unique since there are many ways to reach one and the same non-relativistic limit (simplicity and convenience being the first considerations, though eventually it may be desirable to explore a range of possibilities).

In this way the non-relativistic HOOP equation of motion is generalized into a relativistic HOOP, with a full panoply of ten conservation laws going with the tenfold Poincaré symmetry. The final stage is to convert the relativistic HOOP into Hamiltonian form, employing the classic scheme of Ostrogradsky [5], and then into quantum theory via canonical commutation rules. A novel feature of Hamiltonian HOOP, whether relativistic or non-relativistic, is the occurrence of both position and velocity of the single particle as generalized coordinates, with conjugate momenta for each, instead of the (ZIT untenable) coordinates and momenta for the original individual two particles.

It may be noticed that Gaida *et al* [6] have considered relativistic Lagrangians for a system or many particles based on all of the particle coordinates and their time derivatives. Their finding is that for interacting particles derivatives of all orders, up to infinite orders, must be included. In contrast, in the present scheme of *single-particle* dynamics, quite finite orders of derivatives are adequate both for Poincaré covariance and for interaction, reflected in the twisting and turning of the single worldline. The full complement of conservative laws (linear momentum, angular momentum, energy, uniform centre-of-mass motion) may be brought forth without difficulty [4]; they give Poisson brackets in conformity with the Lie algebra of the Poincaré group, and worldline invariance is obtained.

2. The HOOP oscillator

Starting with the non-relativistic equations of motion for a pair of particles coupled by a spring,

$$m_1\ddot{\boldsymbol{r}}_1 = -k(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

$$m_2\ddot{\boldsymbol{r}}_2 = k(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

one may decouple directly by solving the first equation for $\boldsymbol{r}_2$ and introducing it into the second one, giving the fourth-order HOOP equation

$$\frac{m_1 m_2}{k}\boldsymbol{r}_1^{(4)} + (m_1 + m_2)\ddot{\boldsymbol{r}}_1 = 0. \tag{1}$$

Dropping the subscript 1 and writing $\boldsymbol{r}_1$ as $\boldsymbol{r}$, $\dot{\boldsymbol{r}}_1$, as $\boldsymbol{v}$ and $\ddot{\boldsymbol{r}}_1$ as $\boldsymbol{a}$, a Lagrangian, giving equation (1) as Euler–Lagrange equations, is

$$L_0 = -\frac{1}{2}\frac{\mu}{c^2}\boldsymbol{v}^2 + \lambda\boldsymbol{a}^2 \tag{2}$$

where, with relativistic forethought,

$$\mu \equiv (m_1 + m_2)c^2 \qquad \lambda \equiv \frac{1}{2}\frac{m_1 m_2}{k}.$$

Going now to Ostrogradsky's Hamiltonian [5] scheme, we introduce the canonical coordinates

$$\boldsymbol{q} = \boldsymbol{r} \qquad \boldsymbol{Q} = \boldsymbol{v}$$

and their conjugate momenta

$$\boldsymbol{p} = \frac{\partial L_0}{\partial \boldsymbol{v}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \boldsymbol{a}} = -\frac{\mu}{c^2}\boldsymbol{v} - 2\lambda\dot{\boldsymbol{a}}$$

$$\boldsymbol{P} = \frac{\partial L_0}{\partial \boldsymbol{a}} = 2\lambda\boldsymbol{a}$$

to yield the Hamiltonian

$$H_0 = \boldsymbol{p}\cdot\boldsymbol{v} + \boldsymbol{P}\cdot\boldsymbol{a} - L$$

wherein $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{P}) = \boldsymbol{P}/2\lambda$. Thus

$$H_0 = \boldsymbol{p}\cdot\boldsymbol{Q} + \frac{1}{4\lambda}\boldsymbol{P}^2 + \frac{1}{2}\frac{\mu}{c^2}\boldsymbol{Q}^2.$$

Since $\boldsymbol{q}$ is cyclic, $\boldsymbol{p}$ is conserved, being in fact the ordinary linear momentum, upon returning to primitive variables,

$$\boldsymbol{p} = -(m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2).$$

In the reference frame for which $\boldsymbol{p} = 0$, the Hamiltonian is simply

$$H_0 = \frac{1}{4\lambda}\boldsymbol{P}^2 + \frac{1}{2}\frac{\mu}{c^2}\boldsymbol{Q}^2.$$

The correspondence between the variables $\boldsymbol{r}$, $\boldsymbol{v}$, $\boldsymbol{a}$, $\dot{\boldsymbol{a}}$ of the Lagrangian scheme and the familiar conventional variables $\boldsymbol{r}_1$, $\boldsymbol{v}_1$, $\boldsymbol{r}_2$, $\boldsymbol{v}_2$ of the two-particle dynamics is readily obtained using the primary equations of motion:

$$\boldsymbol{r}_2 = \boldsymbol{r}_1 + \frac{m_1}{k}\boldsymbol{a}_1 = \boldsymbol{r} + \frac{m_1}{k}\boldsymbol{a}$$

$$\dot{\boldsymbol{r}}_2 = \boldsymbol{v}_2 = \boldsymbol{v}_1 + \frac{m_1}{k}\dot{\boldsymbol{a}}_1 = \boldsymbol{v} + \frac{m_1}{k}\dot{\boldsymbol{a}}.$$

In a similar vein, the quartet of Ostrogradsky variables $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Q}, \boldsymbol{P}$ may be elaborated in terms of the primitive quartet $\boldsymbol{r}_1, \dot{\boldsymbol{r}}_1, \ddot{\boldsymbol{r}}_1, \dddot{\boldsymbol{r}}_1$ through use of the Hamiltonian equations of motion:

$$\boldsymbol{r}_1 \equiv \boldsymbol{q}$$

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{q}} = \frac{\partial H_0}{\partial \boldsymbol{p}} = \boldsymbol{Q}$$

$$\ddot{\boldsymbol{r}}_1 = \ddot{\boldsymbol{q}} = \dot{\boldsymbol{Q}} = \frac{\partial H_0}{\partial \boldsymbol{P}} = \frac{1}{2\lambda}\boldsymbol{P}$$

$$\dddot{\boldsymbol{r}}_1 = \dddot{\boldsymbol{q}} = -\frac{1}{2\lambda}\frac{\partial H_0}{\partial \boldsymbol{Q}} = -\frac{1}{2\lambda}\left(\boldsymbol{p} + \frac{\mu}{c^2}\boldsymbol{Q}\right).$$

A similar relativistic calculation using the relativistic H below may also be performed, while the recovery of the motion of particle 2 requires a detour that is under investigation.

The relativistic elevation of the foregoing is now very simple. Momentarily using the proper time

$$\mathrm{d}\tau = \gamma^{-1}\,\mathrm{d}t$$

$$\gamma \equiv (1-\beta^2)^{1/2} \qquad \beta \equiv v/c$$

we have from $x_i = (x_1, x_2, x_3, x_4 \equiv \mathrm{i}ct)$, the familiar 4-velocity

$$u_i = \frac{\mathrm{d}x_i}{\mathrm{d}\tau} \qquad \text{or} \qquad (\gamma \boldsymbol{v}, \gamma \mathrm{i}c)$$

and 4-acceleration

$$w_i = \frac{\mathrm{d}u_i}{\mathrm{d}\tau} \qquad \text{or} \qquad \gamma^4(\gamma^{-2}\boldsymbol{a} + \boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{a}, \mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{a})$$

allowing the step-up term by term of the non-relativistic $L_0\,\mathrm{d}\tau$ in equation (2) to the Lorentz scalar

$$\mu\,\mathrm{d}\tau + \gamma w_\alpha w_\alpha\,\mathrm{d}\tau$$

so that, in coordinate time the fully relativistic

$$L\,\mathrm{d}t = \mu\gamma^{-1}\,\mathrm{d}t + \lambda[\gamma^3\boldsymbol{a}^2 + \gamma^5(\boldsymbol{\beta}\cdot\boldsymbol{a})^2]\,\mathrm{d}t$$

correctly encompasses $L_0\,\mathrm{d}t$ in the non-relativistic limit.

The Ostrogradsky machinery now produces

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{a}}$$

$$\boldsymbol{P} = \frac{\partial L}{\partial \boldsymbol{a}} = 2\lambda(\gamma^3\boldsymbol{a} + \gamma^5\boldsymbol{Q}(\boldsymbol{Q}\cdot\boldsymbol{a})^2)$$

as the canonical conjugates of the coordinates $\boldsymbol{q} = \boldsymbol{r}$ and $\boldsymbol{Q} = \boldsymbol{v}$. Inverting the latter to obtain $\boldsymbol{a}(\boldsymbol{P})$

$$\boldsymbol{a} = \frac{\gamma^{-3}}{2\lambda}\left(\boldsymbol{P} - \frac{\boldsymbol{Q}\boldsymbol{Q}\cdot\boldsymbol{P}}{c^2}\right)$$

so that the relativistic Ostrogradsky Hamiltonian $\boldsymbol{p}\cdot\boldsymbol{Q} + \boldsymbol{P}\cdot\boldsymbol{a} - L$ becomes

$$H = \boldsymbol{p}\cdot\boldsymbol{Q} - \mu\gamma^{-1} + \frac{1}{4\lambda}\gamma^{-3}\left(\boldsymbol{P}^2 - \frac{(\boldsymbol{P}\cdot\boldsymbol{Q})^2}{c^2}\right).$$

This simplifies, in the zero-momentum frame $\boldsymbol{p} = 0$ where the angular momentum $\boldsymbol{q}\times\boldsymbol{p}+\boldsymbol{Q}\times\boldsymbol{P}$ is then just $\boldsymbol{L} = \boldsymbol{Q}\times\boldsymbol{P}$ and $(\boldsymbol{Q}\cdot\boldsymbol{P})^2 = \boldsymbol{Q}^2\boldsymbol{P}^2 - \boldsymbol{L}^2$, to

$$H = -\mu\gamma^{-1} + \frac{\gamma^{-5}}{4\lambda}\boldsymbol{P}^2 + \frac{\gamma^{-3}}{4\lambda^2}L^2. \tag{3}$$

The classical motion may be explicitly worked out [7] in terms of elliptic functions.

3. Quantization

Writing the middle term in equation (3) as $f(\boldsymbol{Q})\boldsymbol{P}^2$ and using standard canonical commutation rules $(\boldsymbol{P}_i, \boldsymbol{Q}_j) = \mathrm{i}\hbar\delta_{ij}$, we are faced at once with the ancient (and still unresolved) problem of quantal operator ordering. Two obvious choices of ordering giving Hermitian operators are

$$\boldsymbol{P}\cdot f\boldsymbol{P} \qquad \text{and} \qquad \tfrac{1}{2}(f\boldsymbol{P}^2 + \boldsymbol{P}^2 f)$$

so a quite general spectrum of possibilities for the quantal representation of $f\boldsymbol{P}^2$ is the superposition

$$\alpha(\boldsymbol{P}\cdot f\boldsymbol{P}^2) + \tfrac{1}{2}\beta(f\boldsymbol{P}^2 + \boldsymbol{P}^2 f) \qquad (\alpha+\beta = 1)$$

or

$$f\boldsymbol{P}^2 - \mathrm{i}\hbar(\nabla_Q f)\cdot\boldsymbol{P} - \tfrac{1}{2}\beta\hbar^2\nabla_Q^2.$$

This encompasses several standard ordering rules: the symmetrization, Born–Jordan and Weyl–McCoy rules correspond to $\beta = 1, \frac{2}{3}, \frac{1}{2}$, respectively. Below, based on numerical studies, the choice $\beta = 2$ will be used, since the sensitivity of the eigenvalues of the Hamiltonian operator upon increments of the precise value of β appears to be reduced for β in the vicinity of $\beta = 2$.

Using spherical polar coordinates in the velocity space $\boldsymbol{R}$,

$$|\boldsymbol{Q}| = R \qquad \boldsymbol{P} = -\mathrm{i}\hbar\nabla_R$$

$$\gamma = (1 - R^2/c^2)^{-1/2}$$

$$\nabla_R^2 = \frac{1}{R^2}\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R}R^2 + \frac{1}{R^2}\left(-\frac{\boldsymbol{L}^2}{\hbar^2}\right)$$

the Hamiltonian operator becomes

$$H = -\mu\gamma^{-1} + \frac{1}{4\lambda}\left[5\frac{\hbar^2}{c^2}\gamma^{-3}R\frac{\partial}{\partial R} - \hbar^2\gamma^{-5}\nabla^2 + \frac{15\hbar^2}{2c^2}\beta\gamma^{-1}\left(1 - 2\frac{R^2}{c^2}\right) + \frac{1}{c^2}\gamma^{-3}L^2\right].$$

It is to be emphasized that the radial 'coordinate' $\boldsymbol{R}$ is an Ostrogradsky coordinate representing not position but velocity: the wave equation $H\Psi = E\Psi$ is a wave equation in velocity space: the domain of R/c is $0 \leqslant R/c \leqslant 1$.

Separating radial from angular coordinates in the usual way yields

$$\Psi = F(R)Y_\ell^m(\theta,\phi) \qquad L^2Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m$$

with the radial function F satisfying, after inserting $\gamma(R)$ explicitly, taking units with $c = 1$ and reducing and rearranging

$$\begin{aligned}(1-R^2)^{5/2}\frac{\mathrm{d}^2F}{\mathrm{d}R^2} &+ \left[\frac{2}{R}(1-R^2)^{5/2} - 5R(1-R^2)^{3/2}\right]\frac{\mathrm{d}F}{\mathrm{d}R}\\ &-\left[\ell(\ell+1)\frac{(1-R^2)^{3/2}}{R^2} + \frac{15}{2}\beta(1-2R^2)(1-R^2)^{1/2}\right.\\ &\left.-\frac{2}{W^2}(1-R^2)^{1/2} - \frac{2}{W}\epsilon\right]F = 0 \end{aligned} \tag{4}$$

where

$$\frac{4\lambda c^2\mu}{\hbar^2} \equiv \frac{2}{W^2}$$

$$\frac{4\lambda c^2\mu}{\hbar^2}E \equiv \frac{2}{W^2}\epsilon$$

with $\varepsilon = E/\hbar\omega_0$, $\omega_0 = (k/m)^{1/2}$, $m = m_1 m_2/(m_1 + m_2)$.

It is of interest to compare this result with the non-relativistic one. Using H_0 above (with $\boldsymbol{p} = 0$) one obtains for the radial Schrödinger equation in the Ostrogradsky (velocity) coordinate R (with $c = 1$),

$$\left[\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}R^2\frac{\mathrm{d}}{\mathrm{d}R} - \frac{\ell(\ell+1)}{R^2} - \frac{1}{W^2}R^2 + \frac{2}{W}\epsilon_0\right]F_0 = 0.$$

By rescaling according to

$$\xi = R/\sqrt{W}$$

this is

$$\left[\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{\ell(\ell+1)}{\xi^2} - \xi^2 + 2\epsilon_0\right]F_0 = 0$$

with the well known solutions in Laguerre functions

$$F_0 = \sqrt{\frac{2}{\xi}}L_k^{\ell+1/2}(\xi^2)$$

$$\epsilon_0 = n + \tfrac{3}{2} \qquad (n = 0, 1, 2, \ldots)$$

$$k = \tfrac{1}{2}(n - \ell).$$

Since it is clear that equation (4) is out of reach analytically in terms of known functions, numerical integration is necessary. First, transform away the first derivative using the substitution

$$U(R) = R(1 - R^2)^{5/4}F(R)$$

to give

$$U'' - \phi(R)U = -\frac{2}{W}\epsilon\frac{1}{(1 - R^2)^{5/2}}U$$

$$\phi \equiv \frac{\ell(\ell+1)}{R^2(1 - R^2)} + \frac{15}{2}\beta\frac{1 - 2R^2}{(1 - R^2)^2} - \frac{2/W^2}{(1 - R^2)^2} - \frac{15}{2}\frac{1}{(1 - R^2)} + \frac{5}{4}\frac{R^2}{(1 - R^2}.$$

Now discretize, following a well known integration method [8], by placing R at the lattice points $R_0, R_1, \ldots R_j, \ldots (R_j = jh)$ with $U(R_j) \equiv U_j$ and $\phi(R_j) \equiv \phi_j$, while U'' is approximated as

$$U'' \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}.$$

Thereupon

$$-U_{j+1} + (2 + h^2\phi_j)U_j - U_{j+1} = \frac{1}{(1 - R_j^2)^{5/2}}\frac{2h^2}{W}\epsilon U_j$$

so that we have a system of linear equations in $U_0, U_1, U_2, \ldots,$ and an eigenvalue problem therein, which may be handled by well known techniques [9].

Representative solutions are given in figures 3–6, where relativistic (R) and non-relativistic (NR) radial functions $F(\xi)$ and $F_0(\xi)$ are shown up to $n = 3$, for $\beta = 2$, $W = 0.1$ (note that very small $W \ll 1$ corresponds to the nearly non-relativistic limit, while $W \gg 1$ corresponds to the highly relativistic case). For $W = 0.1$ the situation is borderline non-relativistic, with eigenfunctions that are crudely comparable to the non-relativistic case yet readily distinguishable from it.

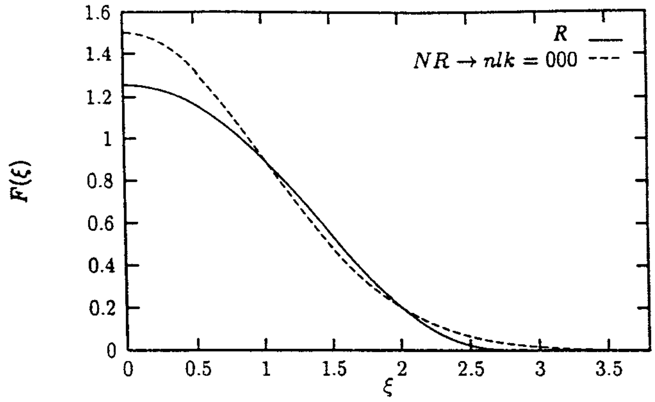

Figure 3. Comparing relativistic and non-relativistic radial wavefunctions: $n = 0$.

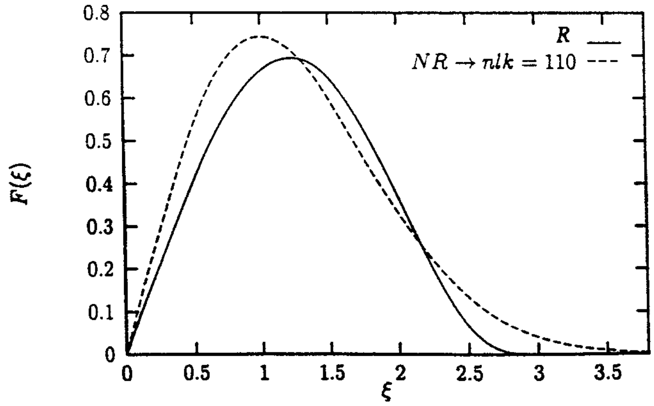

Figure 4. Comparing relativistic and non-relativistic radial wavefunctions: $n = 1$.

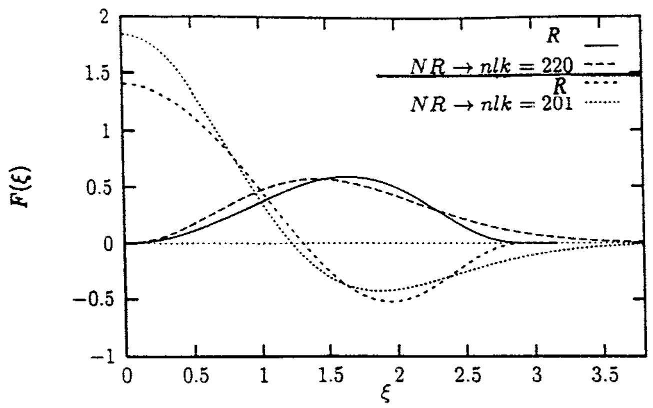

Figure 5. Comparing relativistic and non-relativistic radial wavefunctions: $n = 2$.

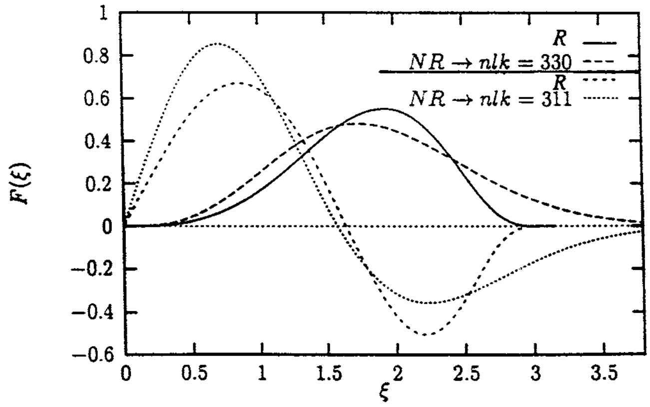

Figure 6. Comparing relativistic and non-relativistic radial wavefunctions: $n = 3$.

4. Conclusion

We may summarize as follows. In a procedure that is completely non-perturbative, the relativistic oscillator has been presented as a single-particle equation of motion of higher (fourth) order, which generalizes the non-relativistic pair of equations of motion of two particles coupled by a spring. The higher-order dynamics has been Hamiltonized according to Ostrogradsky's classical method and, in the centre-of-momentum frame, a Schrödinger equation in velocity space has been produced and analysed fully, down to a numerical solution of the appropriate radial wave equation. This illustrates, in concrete detail, the HOOP approach to the many-particle relativistic problem.

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